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THE STABILITY OF MECHANICAL SYSTEMS SUBJECTED TO IMPULSIVE ACTIONS†

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Mechanical systems subjected to an impulsive load at set times are considered. The impulsive forces depend on generalized coordinates and cause variation of the generalized velocities of the system. Equations of this type describe the vibrations of structures due to seismic shocks, the dynamics of systems of rigid bodies on a moving train or a landing aircraft, etc. The instants of the impulsive action may have limit points, as in the case of shocks which attenuate in a geometric progression. Various problems related to the stability of motion will be discussed. First, general properties of solutions with infinitely many impulse times will be established, namely, their existence, uniqueness and the nature of their dependence on the parameters and initial conditions. The results obtained enable in particular, the linearization method to be used to investigate the stability. Particular attention is paid to non-linear Hamiltonian systems with generalized (impulsive) potential. It is shown that such systems possess a cononical phase flow, and KAM-theory may be used to investigate the stability. An important part of such investigations is the problem of constructing the stability domain in the first approximation, the solution of which frequently involves an analysis of Hill's equation. A series of sufficient conditions are obtained for the stability of the trivial solution of Hill's equation with periodic shocks, generalizing well-known criteria which are applicable to smooth systems. The example of a pendulum whose suspension point is given periodic equal vertical impulses is considered in detail. © 2001 Elsevier Science Ltd. All rights reserved.

Previously, systems subjected to impulsive actions have been considered on the assumption that there art no beats [1–3]; it has been shown that the characteristic equation for a linear Hamiltonian system is reciprocal and that stability is only possible in critical cases.

1. THE GENERAL PROPERTIES OF THE SOLUTION

Consider the system

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right)-\frac{\partial T}{\partial \mathbf{q}}=\mathbf{Q}(t,\mathbf{q},\dot{\mathbf{q}})+\sum_{\alpha\in A}\mathbf{I}_\alpha(\mathbf{q})\delta(t-\tau_\alpha),\quad \mathbf{q}\in R^n \tag{1.1}$$

where $\mathbf{q}$ are generalized coordinates, T is the kinetic energy, $\mathbf{Q} \in R^n$ are generalized forces, I_α are impulses applied to the system at set times τ_α in some denumerable set A, and δ is the Dirac function. The presence of impulsive actions causes abrupt changes in the phase variables in accordance with the formulae

$$\mathbf{q}(\tau_\alpha+0)=\mathbf{q}(\tau_\alpha-0),\quad \frac{\partial T}{\partial \dot{\mathbf{q}}}(\tau_\alpha+0)=\frac{\partial T}{\partial \dot{\mathbf{q}}}(\tau_\alpha-0)+I_\alpha \tag{1.2}$$

The set A may have limit points $\tau_1^*, \tau_2^*, \ldots$, in which case it is a finite or denumerable union of monotone increasing sequences, one of which may be unbounded, while the others tend to $\tau_1^*, \tau_2^*, \ldots$, (it will be assumed that the sequence $\{\tau_k^*\}$ itself is finite or increases without limit and that the numbers τ_k^* do not belong to A).

We will first define what we mean by a solutions of system (1.1).

Definition 1. A function $\mathbf{q}(t)$ will be called a solution of system (1.1), (1.2) in a time interval J, with abrupt changes (jumps) at time $\{\tau_k\}$, if:

1) $\mathbf{q}(t)$ is continuous in J and twice differentiable at $t \neq \tau_k$, and Eqs (1.1) are satisfied;
2) the jumps are described by formulae (1.2);

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3) if τ^* is a limit point of the sequence of jump times but not itself a jump time, then the derivative $\dot{\mathbf{q}}(t)$ is continuous at $t = \tau^*$.

The last property does not follow directly from the equations of motion, but it is essential in order to be able to extend the solution in a unique manner to values of $t > \tau^*$. In mechanical terms, it means that there is no impulsive actions at times not in the set A. In the case $\tau^* \in$ A, the jump is described by formulae (1.2).

We specify a domain $\Omega_1 = J_1 \times \bar{D} \times \bar{K}^n$ in the extended phase space of system (1.1), where $J_1 = [t_0, t_1]$, $t_1 \in (\tau_1^*, \tau_2^*)$, $\bar{D}$ is a compact domain in the configuration space, and $\bar{K}^n = \{|\dot{q}_j| \leqslant K, j = 1, 2, \ldots, n$ is a cube in the space of generalized velocities.

Proposition 1. We assume that

1) the kinetic energy T and generalized forces $\mathbf{Q}$ are described by functions which are continuously differentiable in the domain Ω_1;

2) in the domain $\bar{D}$, the impulses I_k are uniformly majorized by some convergent sequence of real numbers, that is, for any $\mathbf{q} \in \bar{D}$ we have

$$\|I_k(\mathbf{q})\| \leqslant a_k,\ a_1 + a_2 + \ldots < \infty$$

Then system (1.1) has a unique solution for any initial conditions $\mathbf{q}(t_0) \in D$, $\dot{\mathbf{q}}(t_0) \in K^n$. This solution may be continued up to the boundary of the domain Ω_1.

Proof. The general properties of solutions with a finite number of impulsive actions have been established previously [1, 2]: a unique solution exists which reaches the boundary of the domain Ω_1 earlier than the time τ_1^*. It therefore remains to consider the case of solutions which includes an infinite sequence of impulses.

Let $\mathbf{u}_N(t) = (\mathbf{q}(t), \dot{\mathbf{q}}(t))_N$ denote a solution of system (1.1) with the given initial data, which experiences impulsive actions (1.2) only at time $\tau_1, \tau_2, \ldots, \tau_N$. Then, by virtue of our assumptions, $\{\mathbf{u}_N(\tau^* - 0)\}_{N=1}^{\infty}$ is a fundamental sequence in R^{2n}. In the limit as $N \to \infty$ we obtain a solution $\mathbf{u}(t) = (\mathbf{q}(t), \dot{\mathbf{q}}(t))$ in the interval (t_0, τ_1^*), which may be extended to $t \in (\tau_1^*, t_1)$ in a unique manner, by virtue of the given definition of a solution, which it was required to prove.

Corollary. The solution constructed may be extended in an analogous manner to $t \in (\tau_2^*, \tau_3^*)$, etc., provided that the majorization condition 2 is satisfied for each of the sequences of impulses.

We will now examine how solutions with infinitely many jumps depend on the parameters or initial conditions. Suppose the functions T, $\mathbf{Q}$ and I_α in Eq. (1.1) depend on a parameter $\mu \in R^s$, where s is some natural number.

Proposition 2. Assume that

1) the kinetic energy T and generalized forces $\mathbf{Q}$ are described by functions which are continuously differentiable in the domain $\Omega_1 \times M$, where M is a domain in the parameter space, and the function $I_k(\mathbf{q}, \mu)$ are continuously differentiable in the domain $\bar{D} \times M$;

2) in the domain $\bar{D} \times M$ the impulses I_k are uniformly majorized by some convergent sequence of real numbers, that is, for any $\mathbf{q} \in \bar{D}$, $\mu \in M$, we have $\|I_k(q, \mu)\| \leqslant a_k$, $a_1 + a_2 + \ldots < \infty$;

3) in the domain $\bar{D} \times M$, the Jacobians $(\partial I_k/\partial\mu)$ are uniformly majorized by a convergent sequence of real numbers, that is, for any $\mathbf{q} \in \bar{D}$, $\mu \in M$, we have $\|\partial I_k(\mathbf{q}, \mu)/\partial\mu\| \leqslant b_k$, $b_1 + b_2 + \ldots < \infty$, for some norm in the space of $(n \times s)$ matrices.

Then the solutions of system (1.1) in the domain $\Omega \times M$ are continuously differentiable functions of the parameters. In addition, for values of t such that the solution is defined, the derivative $\partial\mathbf{u}(t)/\partial\mu = (\partial\mathbf{q}(t)/\partial\mu, \partial\dot{\mathbf{q}}(t)/\partial\mu)$ is the limit of the sequence of derivatives $\partial\mathbf{u}_N(t)/\partial\mu$ as $N \to \infty$ (the function $\mathbf{u}_N(t)$ were defined in the proof of Proposition 1).

The proof of this proposition is analogous to that of Proposition 1; it uses the theorem on the differentiability of a sequence of mappings [4].

Corollaries. 1. Defining the deviations of the initial data of Eq. (1.1) from certain fixed values as parameters, we obtain a theorem according to which solutions are differentiable with respect to the initial data.

2. Proposition 2 can be generalized to the case of higher-order derivatives on the basis of the corresponding result for a sequence of mappings [4]. If a majorization condition of type 3 is valid not only for the first derivative but for all derivatives up to order m inclusive, then the Cauchy problem has

an m times differentiable solution, whose derivatives of orders $1, \ldots, m$ for $t > \tau^*$ are the limits of the corresponding derivatives of the sequence $\{u_N(t)\}$.

Let us assume now that $\mathbf{q}^\circ(t)$ is a solution of system (1.1) defined in an unbounded interval $t > t_0$. The usual definitions of stability of this solution in Lyapunov's sense and asymptotic stability may now be formulated. If the conditions of Propositions 1 and 2 are satisfied, the linearization method may be used to investigate stability. Special care is necessary in the case when the instants of impulsive action have a limit point $\tau^* < \infty$, because of the need to evaluate infinite products of Jacobians. The convergence of these products may be established using Proposition 2.

2. HAMILTONIAN SYSTEMS WITH IMPULSIVE ACTIONS

An important special case of mechanical systems is described by equations of the form

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}}, \quad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}}$$

where $\mathbf{p} = \partial T/\partial \dot{\mathbf{q}} \in R^n$ are generalized impulses and $H = H(t, \mathbf{q}, \mathbf{p})$ is the Hamiltonian, which satisfies certain smoothness conditions. If a generalized function $\sum_{\alpha \in A} U_\alpha(\mathbf{q})\delta(t-\tau_\alpha)$ is added to the Hamiltonian, where the functions $U_\alpha(\mathbf{q})$ are differentiable, we obtain a canonical system with impulsive actions.

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}}, \quad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}} + \sum_{\alpha \in A} \mathbf{I}_\alpha(\mathbf{q})\delta(t-\tau_\alpha), \quad \mathbf{I}_\alpha = \operatorname{grad} U_\alpha \tag{2.1}$$

In a Hamiltonian system, the transformation of phase variables effected by the phase flow is a canonical transformation [5]. This means that, at any time $t_k > t_0$, the relation between the initial values $(\mathbf{q}_0, \mathbf{p}_0)$ of the phase variables and their values $(\mathbf{q}_k, \mathbf{p}_k)$ at time $t = t_k$ may be expressed in terms of a certain generating function $S(t_k, \mathbf{q}_0, \mathbf{p}_k)$ by the formulae

$$\mathbf{q}_k = \partial S/\partial \mathbf{p}_k, \quad \mathbf{p}_0 = \partial S/\partial \mathbf{q}_0$$

Proposition. The phase flow of system (2.1) defines a canonical transformation for $t \neq \tau_\alpha$ $(\alpha \in A)$.

The proof of this proposition reduces to verifying the canonicity of the impulsive transformation of the phase variables at the jump times τ_α, which is given by the formulae

$$\mathbf{q}(\tau_\alpha + 0) = \mathbf{q}(\tau_\alpha - 0), \quad \mathbf{p}(\tau_\alpha + 0) = \mathbf{p}(\alpha_\alpha - 0) + \mathbf{I}_\alpha(\mathbf{q}(\tau_\alpha)) \tag{2.2}$$

As is easily verified, the transformation (2.2) may be defined in terms of a generating function

$$S_\alpha = \mathbf{q}(\tau_\alpha - 0)\mathbf{p}(\tau_\alpha + 0) + U_\alpha(\tau_\alpha - 0)$$

Let us assume that the interval (t_0, t_k) contains a finite number of jump times $\tau_1, \tau_2, \ldots, \tau_S$. Then the transformation Π_k: $(q_0, p_0) \to (q_k, p_k)$ may be represented in the form of a composition of canonical mappings

$$\Pi_k = N(\tau_s, t_k) \circ U_s \circ N(\tau_{s-1}, \tau_s) \circ \ldots \circ U_1 \circ N(t_0, \tau_1) \tag{2.3}$$

where U_j is the transformation (2.2) with $\alpha = j$, and $N(a, b)$ is the transformation along the phase flow of the regular part of system (2.1) between times $t = a + 0$ and $t = b - 0$. Since the canonical transformations form a group, formula (2.3) defines a mapping of the same type.

Now assume that the interval (t_0, t_k) contains an infinite sequence of jump points $\tau_j \to \tau^*$, and that the conditions of Proposition 1 are satisfied. Then the product (2.3) is infinite:

$$\Pi_k = N(\tau^*, t_k) \circ \ldots \circ U_s \circ N(\tau_{s-1}, \tau_s) \circ \ldots \circ U_1 \circ N(t_0, \tau_1) \tag{2.4}$$

We shall prove that this product converges to some canonical mapping. If the partial derivatives of the Hamiltonian in the relevant domain of the phase space Ω_1 are bounded by a constant M, then the mapping $N(\tau_{j-1}, \tau_j)$ is close to an identity, that is, we have a limit

$$\| \mathbf{x} - N(\tau_{j-1}, \tau_j)(\mathbf{x}) \| \leqslant M(\tau_j - \tau_{j-1})\sqrt{2n}, \quad \mathbf{x} \in \Omega \tag{2.5}$$

where $\|\cdot\|$ denotes the Euclidean norm. In addition, the mapping U_s is also close to an identity, meaning that

$$\|\mathbf{x} - U_s(\mathbf{x})\| = \|I_s(\mathbf{x})\| \leqslant a_s \tag{2.6}$$

Conditions (2.5) and (2.6) guarantee that the product (2.4) will be uniformly convergent, since

$$a_1 + a_2 + \ldots < \infty \text{ и } (\tau_2 - \tau_1) + (\tau_3 - \tau_2) + \ldots = \tau^* - \tau_1 < \infty$$

To verify that the limit mapping is canonical, one can use a characteristic property of canonical mappings, such as the preservation of integrals of the form **pdq** along closed contours [5], which is conserved on taking the limit.
Consequently, formula (2.4) defines a canonical mapping.

Remark. In [1, 2], linear Hamiltonian systems with impulsive actions of a more general form were considered: all the phase variables were allowed to vary at the jumps. In the mechanical systems under discussion here, the generalized coordinates are continuous functions of time.

We will now discuss the question of the stability of the equilibrium positions and the periodic solutions of system (2.1). Corresponding to solutions of both these types we have fixed points of the Poincaré mapping. A fixed point of a canonical mapping cannot be asymptotically stable, since the characteristic equation is reciprocal. A necessary condition for stability is that all roots of the characteristic equation equal unity in absolute value [5]. Sufficient conditions for stability in the case $n = 1$ are determined by the Arnol'd–Moser theorem [5]: to verify them, one must verify that the normal form of the mapping is non-degenerate in the neighbourhood of the fixed point.

Let us assume that the Hamiltonian in system (2.1) is periodic in time with period τ, while the set of instants of impulsive actions is invariant to translation by τ. If the functions H and $\mathbf{I}_\alpha$ are sufficiently smooth, the Arnol'd–Moser theorem may be used to investigate the stability of equilibrium positions and periodic solutions of system (2.1).

Example. Consider a mathematical pendulum whose suspension point is subject to equal shock impulses applied periodically along the vertical. We choose units of measurement such that the length of the pendulum, its mass and the acceleration due to gravity are equal to unity. Then the equations of motion will be

$$\ddot{x} + \sin x = I \sum_{j=-\infty}^{\infty} \delta(t - j\tau) \sin x \tag{2.7}$$

where x is the angle of deviation from the vertical, τ is the time interval between successive impulses and I is the magnitude of the impulse. System (2.7) has equilibrium positions $x = 0$ and $x = \pi$ (the lower and upper positions of the pendulum): at these points the moments of the gravity force and of the impulsive forces vanish. We will first study the stability of the lower equilibrium point. To do this, we must construct the Poincaré mapping along the phase flow in time τ.

When there are no impulses, Eqs (2.7) can be integrated in terms of Jacobi elliptic functions [6]. In the neighbourhood of the lower equilibrium position the motion with initial data $x = 0$, $\dot{x} = 2k$ at $t = 0$ is described by the formulae

$$x = 2\arcsin(k \operatorname{sn} t), \quad \dot{x} = 2k \operatorname{cn} t \tag{2.8}$$

where $\operatorname{sn} t$ and $\operatorname{cn} t$ are the elliptic sine and cosine with modulus k. Solution (2.8) can be generalized to arbitrary initial data, using the general properties of elliptic functions. The result is

$$\sin\frac{x}{2} = \left(\frac{\dot{x}_0}{2}\operatorname{sn} t \cos\frac{x_0}{2} + \operatorname{cn} t \operatorname{dn} t \sin\frac{x_0}{2}\right)\left(1 - \operatorname{sn}^2 t \sin^2\frac{x_0}{2}\right)^{-1} \tag{2.9}$$

$$\dot{x} = (\dot{x}_0 \operatorname{cn} t - \operatorname{sn} t \operatorname{dn} t \sin x_0)\left(1 - \operatorname{sn}^2 t \sin^2\frac{x_0}{2}\right)^{-1}, \quad k^2 = \frac{\dot{x}_0^2}{4} + \sin^2\frac{x_0}{2}$$

The transformation of the phase variables in time τ (Poincaré mapping) is described by the formulae

$$\dot{x}_1 = x(\tau), \quad \dot{x}_1 = \dot{x}(\tau) + I\sin x(\tau) \tag{2.10}$$

The functions $x(\tau), \dot{x}(\tau)$ are defined by (2.9).

For convenience, we will change to canonical variables q, p by the formulae

$$q = 2\sin\frac{x}{2}, \quad \dot{x} = p\cos\frac{x}{2}$$

In these variables, relations (2.9) and (2.10) become

$$q_1 = \left(p_0 \operatorname{sn}\tau\left(1-\frac{q_0^2}{4}\right) + q_0 \operatorname{cn}\tau \operatorname{dn}\tau\right)\left(1-\operatorname{sn}^2\tau\frac{q_0^2}{4}\right)^{-1}$$

$$p_1 = (p_0\operatorname{cn}\tau - q_0\operatorname{sn}\tau\operatorname{dn}\tau)\left(1-\operatorname{sn}^2\tau\frac{q_0^2}{4}\right)^{-1}\left(1-\frac{q_0^2}{4}\right)^{1/2}\left(1-\frac{q_1^2}{4}\right)^{-1/2} + Iq_1 \tag{2.11}$$

$$k^2 = \frac{1}{4}\left(q_0^2 + \left(1-\frac{q_0^2}{4}\right)p_0^2\right)$$

To expand the mapping (2.11) in powers of q_0, p_0 we must first perform that operation for the elliptic functions occurring in it. Using expansions of the latter in Fourier series [7], we obtain

$$\operatorname{sn}\tau = \sin\tau + \frac{k^2}{4}(\sin\tau\cos\tau - \tau)\cos\tau + O(k^4)$$

$$\operatorname{cn}\tau = \cos\tau - \frac{k^2}{4}(\sin\tau\cos\tau - \tau)\sin\tau + O(k^4) \tag{2.12}$$

$$\operatorname{dn}\tau = 1 - \frac{k^2}{4}(1-\cos 2\tau) + O(k^4)$$

Substituting these expansions into (2.11), we obtain

$$q_1 = q_0\cos\tau + p_0\sin\tau + \sum_{r+s=3} \alpha_{rs} q_0^r p_0^s + \dots$$

$$p_1 = q_0(-\sin\tau + I\cos\tau) + p_0(\cos\tau + I\sin\tau) + \sum_{r+s=3}\beta_{rs}q_0^r p_0^s + \dots$$

$$\alpha_{30} = \frac{1}{16}\sin\tau(\sin\tau\cos\tau + \tau), \quad \alpha_{21} = -\frac{1}{16}\cos\tau(3\sin\tau\cos\tau + \tau) \tag{2.13}$$

$$\alpha_{12} = -\frac{1}{16}\sin\tau(3\sin\tau\cos\tau - \tau), \quad \alpha_{03} = \frac{1}{16}\cos\tau(\sin\tau\cos\tau - \tau)$$

$$\beta_{30} = -\alpha_{03} + I\alpha_{30}, \quad \beta_{21} = \alpha_{12} + I\alpha_{21}, \quad \beta_{12} = -\alpha_{21} + I\alpha_{12}, \quad \beta_{03} = \alpha_{30} + I\alpha_{03}$$

Expansion terms of order five and higher are omitted in (2.13).

First let us investigate the linear part of mapping (2.13). The characteristic equation is

$$\rho^2 - 2A\rho + 1 = 0, \quad A = \cos\tau + (I/2)\sin\tau$$

A necessary condition for the lower position of the pendulum to be stable is

$$|A| < 1 \tag{2.14}$$

If inequality (2.14) is reversed, the equilibrium position is unstable. The domain (2.14) in the plane of parameters $\tau > 0, I$ is shown in Fig. 1 (the unhatched area). The boundaries of this domain are the curves $I = 2\,\mathrm{tg}(\tau/2)$ and $\tau = 2m\pi$ ($m \in Z_+$) (in these two cases $\rho_{1,2} = 1$), and also $I = -2\,\mathrm{ctg}\,(\tau/2)$ (the dashed curves) and $\tau = (1+2m)\pi$ (and then $\rho_{1,2} = -1$).

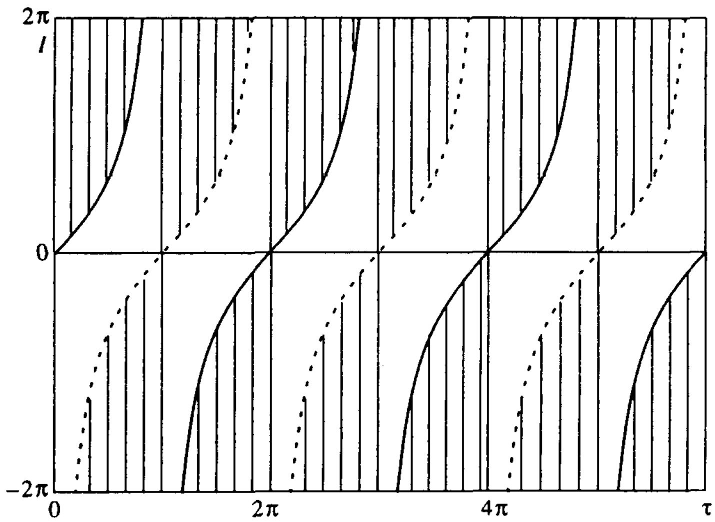

Fig. 1

To solve the problem of stability in the interior of the domain (2.14) in the strictly non-linear sense, we must reduce mapping (2.13) to normal form. We take the canonical linear change of variables in the form

$$q = \frac{1}{\alpha} Q, \quad p = \frac{I}{2\alpha} + \alpha P, \quad \alpha^2 = \frac{\sin\varphi}{\sin\tau}$$

The angle φ is determined from the conditions $\cos\varphi = A$, $\sin\varphi \sin\tau > 0$.

In the new variables, mapping (2.13) is

$$\begin{aligned} Q_1 &= Q_0\cos\varphi + P_0\sin\varphi + \sum_{r+s=3} a_{rs} Q_0^r P_0^s + \ldots, \\ P_1 &= -Q_0\sin\varphi + P_0\cos\varphi + \sum_{r+s=3} b_{rs} Q_0^r P_0^s + \ldots \end{aligned} \tag{2.15}$$

$$a_{30} = \frac{1}{\alpha^2}\left(\alpha_{30} + \frac{I}{2}\alpha_{21} + \frac{I^2}{4}\alpha_{12} + \frac{I^3}{8}\alpha_{03}\right), \quad a_{03} = \alpha^4\alpha_{03}$$

$$a_{21} = \alpha_{21} + I\alpha_{12} + \frac{3}{4}I^2\alpha_{03}, \quad a_{12} = \alpha^2\alpha_{12} + \frac{3}{2}I\alpha^2\alpha_{03}$$

$$b_{30} = \frac{1}{\alpha^4}\left(\beta_{30} + \frac{I}{2}\beta_{21} + \frac{I^2}{4}\beta_{12} + \frac{I^3}{8}\beta_{03}\right) - \frac{I}{2\alpha^2}a_{30}$$

$$b_{03} = \alpha^2\beta_{03} - \frac{I}{2\alpha^2}a_{03}$$

$$b_{21} = \frac{1}{\alpha^2}\left(\beta_{21} + I\beta_{12} + \frac{3}{4}I^2\beta_{03}\right) - \frac{I}{2\alpha^2}a_{21}$$

$$b_{12} = \beta_{12} + \frac{3}{2}I\beta_{03} - \frac{I}{2\alpha^2}a_{12}$$

This mapping contains no quadratic terms. It will therefore suffice to investigate two cases: $\cos\varphi \neq 0$ (the non-resonant case) and $\cos\varphi = 0$ (fourth-order resonance). The condition for the normal form in the non-resonant case to be non-degenerate is [8]

$$3a_{30} + b_{21} + a_{12} + 3b_{03} \neq 0$$

Carrying out the necessary calculations, we obtain

$$3a_{30}+b_{21}+a_{12}+3b_{03}=\frac{1}{16\sin\varphi}(3I^2(\tau-\sin\tau\cos\tau)+8\tau\sin^2\varphi)\neq 0$$

By the Arnol'd–Moser theorem, this implies that the fixed point is stable.

The case of fourth-order resonance occurs on the curve $I = -2\operatorname{ctg}\tau$ in the parameter plane. In this case the inequality

$$(a_{30}+b_{30})(a_{30}+2a_{12}+b_{03})>2(a_{03}-b_{30})^2 \tag{2.16}$$

guarantees stability; if the sign of the inequality is reversed, the fixed points is unstable. Calculations have shown that inequality (2.16) holds in the domain (2.14).

Thus, in this example, inequality (2.14) is not only a necessary but also a sufficient condition for stability.

We will now investigate what happens when the equilibrium position becomes unstable. To do this we consider the fixed points of mapping (2.11) close to the origin. To such points there correspond τ-periodic motions of the pendulum. Since the total mechanical energy of the pendulum is conserved in the inter-shock intervals, it must also be conserved at the shocks in τ-periodic motion. The simplest motions of this kind are described by closed curves in the phase plane which cut the ordinate axis at the shock times. In the interval between successive shocks, the trajectory describes either an integer number of revolutions about the origin (in which case the period is τ) or a semi-integer number (in which case the period is 2τ), whence we obtain

$$\tau=2mK(k),\qquad m\in N$$

where $K(k)$ is the complete elliptic integral of the first kind, expansion of which [7] gives

$$\tau=m\pi(1+\tfrac{1}{4}k^2+O(k^4))$$

Thus, in the plane of the parameters τ, I periodic motions of the type under consideration exist to the right of the vertical straight lines $\tau = m\pi$. Consequently, the parts of these lines in the upper half-plane constitute safe bifurcation boundaries of the stable lower equilibrium position of the pendulum and the parts in the lower half-plane are unsafe bifurcation boundaries.

Motions of the second kind are characterized by abrupt reversal of the velocity at shocks. Motions of this kind are asymmetric: some of them are shown in Fig. 2: (a) $I > 0$, the pendulum does not pass through the lower position; (b) $I > 0$, in the interval between shocks, the pendulum performs a complete oscillation plus a partial one analogous to subcase a); (c) $I < 0$, the pendulum passes through the lower equilibrium position but does not perform a complete oscillation in the inter-shock interval; (d) $I < 0$, over one period the pendulum performs a complete oscillation and one partial one analogous to subcase c). (For clarity, the superimposed parts of the trajectories are shown separately.) Each of the motions listed has a mirror image, obtained by reflecting the phase trajectory in the ordinate axis. There is an infinite set of families of motions of this kind, differing from one another in the number of complete oscillations of the pendulum per period, the sign of the impulse I and the sign of the variable x at the shock times.

Using relations (2.10) and reduction formulae for elliptic functions [7], we obtain for all these families the equality

$$I=2\operatorname{sn}\frac{\tau}{2}\operatorname{dn}\frac{\tau}{2}\left(\operatorname{cn}\frac{\tau}{2}\right)^{-1} \tag{2.17}$$

which may be considered for given τ and I as an equation in the modulus k of the elliptic functions (which is equal to the sine of half the maximum angle by which the pendulum deviates from the vertical in the motion under consideration). For small k, by formulae (2.12), Eq. (2.17) becomes

$$I=2\operatorname{tg}\frac{\tau}{2}\left(1+\frac{k^2}{4}\left(\cos\tau-\frac{\tau}{\sin\tau}\right)+O(k^4)\right) \tag{2.18}$$

Since $|\sin\tau| < \tau$, $|\cos\tau| \leqslant 1$, the coefficient of k^2 in (2.18) has the same sign as $\sin\tau$. Hence we can draw the following conclusion concerning the nature of the boundary $I = 2\operatorname{tg}(\tau/2)$ of the stability domain (2.14) corresponding to the roots $\rho_{1,2} = 1$ of the characteristic equation: if $\sin\tau > 0$ (in which

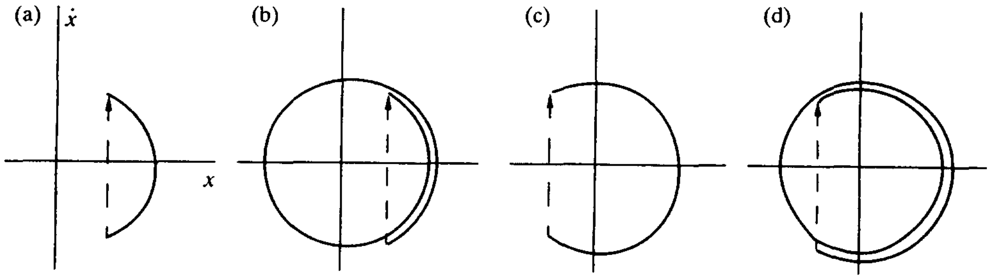

Fig. 2

case $I > 0$ in (2.18)), the bifurcation boundary is unsafe, that is, unstable periodic motions coexist with the stable equilibrium position and disappear when the latter becomes unstable. Conversely, if $\sin \tau < 0$, then $I < 0$ and the bifurcation boundary is safe, that is, stable periodic motions are generated when the equilibrium position becomes unstable ("pitchfork" type bifurcation).

Periodic motions of the third kind are symmetrical and have period 2τ. There is an infinite set of families of such motions, two of which are shown in Fig. 3: (a) $I < 0$ and (b) $I > 0$ (here again, for clarity, superimposed parts of the trajectories are shown separately). Taking (2.8) into consideration, we have the following periodicity condition

$$I = -2\,\mathrm{cn}\frac{\tau}{2}\left(\mathrm{sn}\frac{\tau}{2}\,\mathrm{dn}\frac{\tau}{2}\right)^{-1} \tag{2.19}$$

For small k, using formulae (2.12), we can reduce Eq. (2.19) to the form

$$I = -2\,\mathrm{ctg}\frac{\tau}{2}\left(1 - \frac{k^2}{4}\left(\cos\tau - \frac{\tau}{\sin\tau}\right) + O(k^4)\right) \tag{2.20}$$

From formula (2.20) we can draw the following conclusion as to the nature of the boundary $I = -2\,\mathrm{ctg}\,(\tau/2)$ of the stability domain (2.14) corresponding to roots $\rho_{1,2} = -1$ of the characteristic equation: if $\sin \tau > 0$ (in which case $I < 0$ in (2.20)), the bifurcation boundary is safe (period-doubling bifurcation). Conversely, if $\sin \tau < 0$, then $I > 0$ and the bifurcation boundary is unsafe.

Investigation of the upper equilibrium position of the pendulum proceeds along similar lines. In this paper we will limit ourselves to constructing the domain in which the necessary conditions for stability are satisfied. To do this we linearize system (2.7) in the neighbourhood of $x = \pi$, putting $y = x - \pi$:

$$\ddot{y} - y = I \sum_{j=-\infty}^{\infty} \delta(t - j\tau)$$

The change in the phase variables during a period is described by the formulae

$$y = y_0\,\mathrm{ch}\,\tau + \dot{y}_0\,\mathrm{sh}\,\tau, \quad \dot{y} = y_0\,\mathrm{sh}\,\tau + \dot{y}_0\,\mathrm{ch}\,\tau + I(y_0\,\mathrm{ch}\,\tau + \dot{y}_0\,\mathrm{sh}\,\tau) \tag{2.21}$$

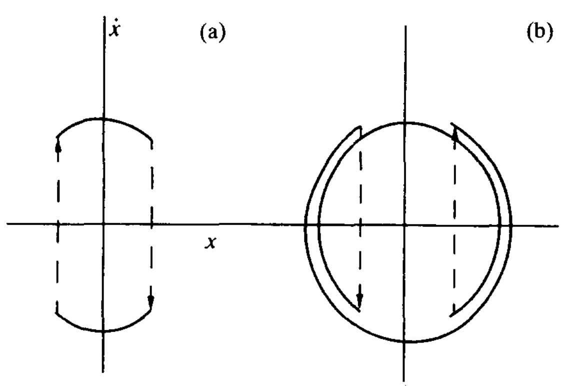

Fig. 3

The condition for stability of the linear mapping (2.21) has the form

$$|2 \operatorname{ch} \tau + I \operatorname{sh} \tau| < 2 \tag{2.22}$$

The domain (2.22) in the parameter plane is constructed in Fig. 4 (the unhatched area); unlike the case considered previously, it is non-periodic. It is interesting to note that this domain contains the whole half-line $I = 2$, $\tau > 0$. Consequently, periodic shocks of intensity $I = 2$ stabilize the upper equilibrium position of the pendulum, except for the dependence on the length of the time interval τ between them. As τ is reduced, the width of the stability domain increases without limit. Analysis shows that the upper boundary of the stability domain (the dashed curve in Fig. 4) is unsafe, and the lower boundary is safe (period-doubling bifurcation).

3. HILL'S EQUATION WITH IMPULSIVE ACTIONS

The foremost problem in investigating the stability of solutions of Hamiltonian systems is to analyse the linear approximation. A case of practical importance is that of the Hill's equation

$$\ddot{q} + f(t)q = 0, \quad f(t+\tau) \equiv f(t) \tag{3.1}$$

Various method have been developed to estimate the characteristic constant of Eq. (3.1) [9], enabling the sufficient conditions for the stability of the trivial equilibrium position to be derived without having to solve a Cauchy problem numerically. One such method is based on estimating the angle of rotation of the solution vector during the time τ. The following proposition has been proved.

Proposition 3 [9]. Let l be a non-negative integer and c a real number such that

$$l\pi/\tau < c < (l+1)\pi/\tau \tag{3.2}$$

Define the functions

$$f_c^- = \min\{f(t), c^2\}, \quad f_c^+ = \max\{f(t), c^2\} \tag{3.3}$$

If

$$l\pi < \frac{1}{c}\int_0^\tau f_c^-(t)dt \leq \frac{1}{c}\int_0^\tau f_c^+(t)dt < (l+1)\pi \tag{3.4}$$

then the trivial solution of Eq. (3.1) is stable.

Let us now assume that system (3.1) experiences impulse actions:

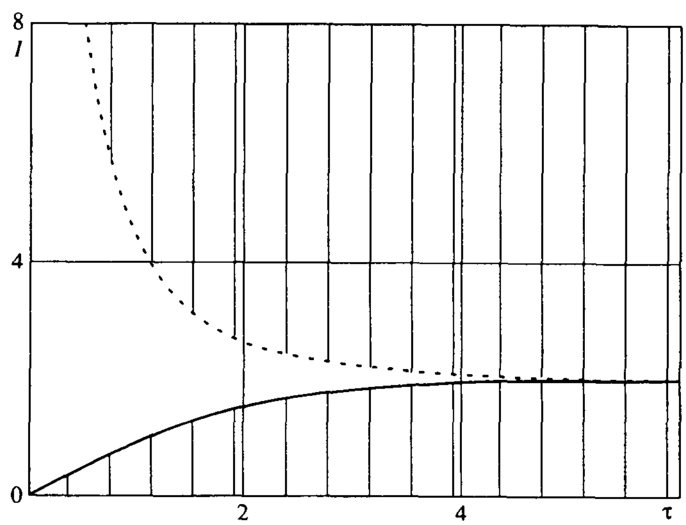

Fig. 4

$$\ddot{q}+\left(f(t)-\sum_{\alpha\in A} I_\alpha\delta(t-\tau_\alpha)\right)q=0 \tag{3.5}$$

where I_α are constants and the set $|\tau_\alpha|$ is invariant to translation by τ.
Proposition 3 admits of the following generalization.

Proposition 4. Let the numbers l and c and the functions $f_c^-(t)$ and $f_c^+(t)$ be defined as in Proposition 3. If

$$l\pi<\frac{1}{c}\left(\int_0^\tau f_c^-(t)dt-S^+\right)\leq\frac{1}{c}\left(\int_0^\tau f_c^+(t)dt+S^-\right)<(l+1)\pi$$
$$S^+=\sum_{\substack{\tau_\alpha\in(0,\tau]\\ I_\alpha>0}} I_\alpha,\quad S^-=-\sum_{\substack{\tau_\alpha\in(0,\tau]\\ I_\alpha<0}} I_\alpha \tag{3.6}$$

then the trivial solution of Eq. (3.5) is stable.

Proof. The integrals (with coefficients $1/c$) in formulae (3.4) and (3.6) are upper and lower limits for the angles of rotation of the vector-solutions of the auxiliary system with Hamiltonian $H = \frac{1}{2}(cp^2 + c^{-1} f(t)q^2)$ (in the phase plane (q, p) these angles $\psi(t)$ are measured in the clockwise direction) [9]. The corresponding auxiliary system with shocks has the Hamiltonian

$$H=\tfrac{1}{2}\left(cp^2+c^{-1}\left(f(t)-\sum I_\alpha\delta(t-\tau_\alpha)\right)q^2\right)$$

Let us estimate the change in angles of rotation at the shocks. Since

$$\operatorname{tg}\psi^-=-p^-/q,\quad \operatorname{tg}\psi^+=-p^+/q$$

(the minus sign is added to take into account the reverse direction in which the angle is measured) and $p^+ = p^- + c^{-1}I_\alpha$, it follows that

$$\operatorname{tg}\psi^+=\operatorname{tg}\psi^- - c^{-1}I_\alpha$$

Applying the formula of finite increments (Lagrange's theorem), we obtain

$$\psi^+=\psi^- - c^{-1}I_\alpha\cos^2\xi,\quad \xi\in(\psi^-,\psi^+)$$

Consequently, positive shocks cause a decrease in the angle of rotation, by an amount not exceeding S^+/c, and negative shocks cause an increase by an amount not exceeding S^-/c. Hence follows the proposition we have formulated.
Various stability criteria [9] based on estimating the integrals

$$\int_0^\tau |f_c^\pm(t)-c^2|\,dt$$

can also be extended to the case of Eq. (3.5).
To that end, it suffices to consider the Dirac functions in Eq. (3.5) as limits of a certain sequence of regular functionals in the space of generalized functions. We finally obtain the following propositions.

Proposition 5. If $l\geq 1$ formula (3.2) and

$$\int_0^\tau (c^2-f_c^-(t))dt+S^+<c(\tau c-l\pi)$$
$$\int_0^\tau (f_c^+(t)-c^2)dt+S^-<2c(l+1)\operatorname{ctg}\frac{\tau c}{2(l+1)} \tag{3.7}$$

or, if $l = 0$

$$\int_0^\tau f(t)dt - S^+ + S^- \geqslant 0,$$
$$\int_0^\tau (f_c^+(t) - c^2)dt + S^- < 2c \operatorname{ctg}\frac{\tau c}{2} \tag{3.8}$$

the trivial solution of Eq. (3.5) is stable.

Proposition 6. If for some $l \in N$

$$\tau\int_0^\tau \left(f(t) - \frac{l^2\pi^2}{\tau^2}\right)dt + \tau S^- \leqslant 2\pi l(l+1)\operatorname{tg}\frac{\pi}{2(l+1)}$$
$$f(t) \geqslant \frac{l^2\pi^2}{\tau^2}, \quad S^+ = 0 \tag{3.9}$$

or

$$\int_0^\tau f_0^+(t)dt + S^- \leqslant \frac{4}{\tau}, \quad \int_0^\tau f(t)dt + S^- - S^+ \geqslant 0 \tag{3.10}$$

(the function f_0^+ was defined in (3.3)), the trivial solution of Eq. (3.5) is stable. (The formula in [9] corresponding to inequalities (3.10) when there are no impulsive forces contains an inaccuracy.)

Remark. If $f(t) \geqslant 0$ and $S^+ = 0$, then the second inequality of (3.10) is automatically valid, while the first may be regarded as an extension of the well-known Lyapunov criterion [9] to systems with impulsive actions.

Proposition 7. If the following conditions are satisfied for some $l \in N$

$$f(t) \leqslant \frac{l^2\pi^2}{\tau^2}, \quad S^- = 0, \quad \tau\int_0^\tau \left(\frac{l^2\pi^2}{\tau^2} - f(t)\right)dt + \tau S^+ < l\pi^2 \tag{3.11}$$

then the trivial solution of Eq. (3.5) is stable.

Example. Consider the linearized equations of motion of a pendulum with periodic impulsive actions

$$\ddot{x} + x = xI\sum_{j=-\infty}^{\infty} \delta(t - j\tau) \tag{3.12}$$

Formula (3.6) with $c = 1$ implies the following conditions

$$\tau - \pi(l+1) < I < \tau - \pi l, \quad l\pi < \tau < (l+1)\pi; \quad l = 0, 1, 2, \ldots \tag{3.13}$$

Since $f(t) = f_0^+(t) \equiv 1$, formulae (3.10) give

$$\tau - 4/\tau \leqslant I \leqslant \tau, \quad \tau \leqslant 2 \tag{3.14}$$

The generalized Lyapunov criterion describes the part of the domain (3.14) lying in the lower half-plane.

Conditions (3.9) in the present case are

$$\tau^2 - l^2\pi^2 - \tau I \leqslant 2\pi l(l+1)\operatorname{tg}\frac{\pi}{2(l+1)}$$
$$I < 0, \quad \tau \geqslant l\pi \tag{3.15}$$

Finally, formulae (3.11) become

$$\tau \leqslant l\pi, \quad I > 0, \quad l^2\pi^2 - \tau^2 + \tau I < l\pi^2 \tag{3.16}$$

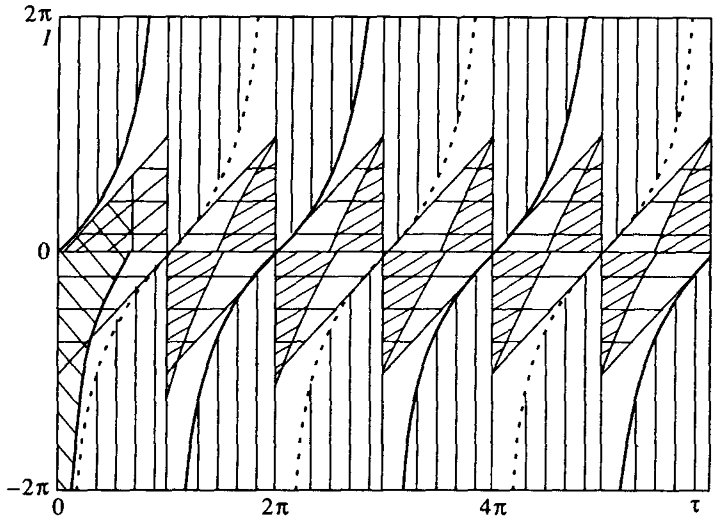

Fig. 5

Figure 5 illustrates the domains (3.13)–(3.16), which constitute part of the stability domain (2.14) in the linear approximation. The horizontal hatching denotes solutions of inequality (3.13) (parallelograms inscribed in each of the stable components), the left-inclined hatching denotes the domain (3.14) (which is unbounded for $\tau < \pi$), and the right-inclined hatching denotes the domains (3.15) and (3.16) (curvilinear triangles in the lower and upper half-planes, respectively).

The advantage of Propositions 4–7 is that they allow sufficient stability conditions to be derived even where a complete construction of the stability domain does not seem possible. Thus, if the pendulum is subject to a periodic series of unequal shocks, its dynamics is described in the first approximation by the equation

$$\ddot{x} + x = x \sum_{\alpha \in A} I_\alpha \delta(t - \tau_\alpha) \tag{3.17}$$

where the sets $\{I_\alpha\}$ and $\{\tau_\alpha\}$ are invariant to the translation of time by τ.

The number of shocks in the interval $(0, \tau]$ may be large or even infinite, but this does not prevent the application of the results described above. In particular, Proposition 4 implies sufficient stability conditions

$$l\pi < \tau - S^+ \leqslant \tau + S^- < (l+1)\pi, \quad l = 0, 1, 2, \ldots \tag{3.18}$$

while formulae (3.10) imply the conditions

$$S^+ \leqslant \tau + S^- \leqslant 4/\tau \tag{3.19}$$

Propositions 6 and 7 are applicable in the case when all the impulses have the same sign. As an example, if the impulses I_α in the interval $(t_0, t_0 + \tau)$ form an infinitely decreasing geometric progression with common ratio $q \in (0, 1)$ and the first term is $I_1 < 0$, then $S^+ = 0$, $S^- = -I_1/(1-q)$. The sufficient conditions (3.18) for stability then take the form

$$l\pi < \tau \leqslant \tau - I_1/(1-q) < (l+1)\pi, \quad l = 0, 1, 2, \ldots$$

and conditions (3.19) become

$$(4/\tau - \tau)(1-q) \leqslant I_1 \leqslant 0$$

In addition, for stability it is sufficient for inequalities (3.15) to hold, where $I = I_1/(1-q)$.

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